A CATALOGUE OF ORIENTABLE 3-MANIFOLDS TRIANGULATED BY 30 COLOURED TETRAHEDRA *

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Abstract

The present paper follows the computational approach to 3-manifold classification via edge-coloured graphs, already performed in [25] (with respect to orientable 3-manifolds up to 28 coloured tetrahedra), in [11] (with respect to non-orientable 3-manifolds up to 26 coloured tetrahedra), in [10] and [4] (with respect to genus two 3-manifolds up to 34 coloured tetrahedra): in fact, by automatic generation and analysis of suitable edge-coloured graphs, called *crystallizations*, we obtain a catalogue of all orientable 3-manifolds admitting coloured triangulations with 30 tetrahedra. These manifolds are unambiguously identified via JSJ decompositions and fibering structures.

It is worth noting that, in the present work, a suitable use of elementary combinatorial moves yields an automatic partition of the elements of the generated crystallization catalogue into equivalence classes, which are proved to be in one-to one correspondence with the homeomorphism classes of the represented manifolds.

Key words: orientable 3-manifold, crystallization, coloured triangulation, complexity.

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1. Introduction

Within the study of the topology of PL-manifolds a great attention has been recently reserved to combinatorial representation methods, enabling to produce and study (possibly with the aid of suitable computer programs) exhaustive catalogues of "small" manifolds, with respect to a given "complexity" criterion: let us recall, as a very significant example, successive results about closed orientable irreducible 3-manifolds whose minimal special spines have increasing number of vertices, up to 11 ([30], [34], [27], [31]), or the analogous studies about closed non-orientable $\mathbb{P}^2$-irreducible 3-manifolds whose minimal special spines have at most 10 vertices ([1], [13], [6], [7], [2], [8], [9]).

During the last thirty years, another representation theory for PL-manifolds has been developed. Its principal feature is generality, i.e. it can represent the whole class of piecewise linear (PL) manifolds, without assumptions about dimension, connectedness, orientability, irreducibility, $\mathbb{P}^2$-irreducibility or boundary properties: see [35], [19], [5], [17], [36], [23], or [3] for a survey on the so-called *crystallization theory*, which makes use of edge-coloured graphs (named also *crystallizations*, under suitable conditions) as a representation tool.

Note that, in virtue of the purely combinatorial nature of the representing objects, crystallization theory turns out to be particularly suitable to computer enumeration. From this view-point, the main existing results concerning the whole class of closed orientable 3-manifolds are described in Lins's book [25], where a catalogue of all orientable 3-manifolds represented by crystallizations with at most 28 vertices is produced and analyzed.[1] On the other hand, [11] takes into account the non-orientable case, while other works deal with restricted classes of 3-manifolds (for example, both orientable and non-orientable euclidean 3-manifolds in [37], genus two orientable 3-manifolds in [10] and [4]...).

The present paper carries on the computational classification of closed orientable 3-manifolds performed in [25], by automatic production and analysis of the complete catalogue of orientable 3-manifolds represented by crystallizations up to 30 vertices (or, equivalently, admitting coloured triangulations with at most 30 tetrahedra). It is worth noting that, in the present work, a suitable use of elementary combinatorial moves yields an automatic partition of the elements of the generated crystallization catalogue $\mathbf{C}^{(30)}$ into equivalence classes, which are proved to be in one-to one correspondence with the homeomorphism classes of the represented manifolds (Proposition 7).

If the attention is restricted to prime 3-manifolds not belonging to the existing Lins's catalogue, the obtained classification may be summarized by the following statement:

Theorem I *There exist exactly forty-one closed connected prime orientable 3-manifolds, which admit a coloured triangulation consisting of 30 tetrahedra and do not admit a coloured triangulation consisting of less than 30 tetrahedra.*
Among them, there are:

- *10 elliptic 3-manifolds;*
- *17 Seifert non-elliptic 3-manifolds (in particular, 2 torus bundles with Nil geometry);*

[1] See also [15], where an unambiguous identification of all elements of Lins's catalogue is given, through JSJ decompositions and fibering structures.

- *2 torus bundles with Sol geometry;*

- *2 manifolds of type* $(K \overset{\sim}{\times} I) \cup (K \overset{\sim}{\times} I)/A \quad (A \in GL(2;\mathbb{Z}),\ \det(A) = -1)$*, with Sol geometry;*

- *7 non-geometric graph manifolds;*

- *3 hyperbolic Dehn-fillings (of the complement of link* 6^3_1*).*

As a consequence of the generation and analysis of catalogue $\mathbf{C}^{(30)}$, it is now possible, for any given bipartite cristallization with at most 30 vertices,[2] to recognize topologically - via computer program DUKE III[3] - the represented manifold, with unambiguous identification by means of JSJ decompositions and fibering structures.

We point out that interesting results follow from a comparative analysis of both complexity and geometric properties of the manifolds represented by the subsequent subsets $\mathcal{C}^{(2p)},\ \ 1 \leq p \leq 15$, of all crystallizations in $\mathbf{C}^{(30)}$ with exactly $2p$ vertices: in fact, for any fixed complexity c, catalogues $\mathcal{C}^{(2p)}$ turn out to cover, for increasing p, first the most "complicated" types of complexity c 3-manifolds and then the simplest ones (see Table 2). As a consequence, catalogues $\mathcal{C}^{(2p)}$, for increasing value of p, appear to be a useful source for interesting examples in order to test conjectures and search for patterns about 3-manifolds.

Finally, the last paragraph of the paper is devoted to present a significant improvement of catalogues $\mathcal{C}^{(2p)}$ (and of the corresponding catalogues $\tilde{\mathcal{C}}^{(2p)}$ for non-orientable 3-manifolds, too): an additional hypothesis on the representing objects yields a considerable reduction of the catalogues without loss of generality as far as the represented 3-manifolds are concerned (see Proposition 11 and Table 3).

2. Basic notions on coloured triangulations of manifolds

As already pointed out, this paper is based on the fundamental tool of *crystallization theory*, i.e. on the possibility of representing PL n-manifolds by means of *edge-coloured graphs* or - equivalently - by means of *coloured triangulations*.

Although crystallization theory extends to manifolds with boundary and several concepts and results hold for singular manifolds too, throughout this paper we will restrict our attention to closed, connected (PL) manifolds.

[2]Non-contracted graphs representing closed orientable 3-manifolds may be handled also in case of a higher number of vertices: see Proposition 9.

[3]Details about the C++ program *DUKE III* for automatic analysis and manipulation of PL-manifolds via edge-coloured graphs may be found on the Web: http://cdm.unimo.it/home/matematica/casali.mariarita/DukeIII.htm

Definition 1. An *(n+1)-coloured graph* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a regular multigraph[4] of degree $n+1$ and $\gamma : E(\Gamma) \to \Delta_n = \{0, 1, \dots, n\}$ is injective on adjacent edges.

The elements of the set $\Delta_n = \{0, 1, \dots, n\}$ are called *colours*; moreover, for each $i \in \Delta_n$, we denote by $\Gamma_{\hat{i}}$ the n-coloured graph obtained from (Γ, γ) by deleting all edges coloured by i.

An n-dimensional pseudocomplex K (see [22] for details) is called *(vertex)-coloured* if it is equipped with a labelling of its vertices by Δ_n, which is injective on each simplex.

The concepts of edge-coloured graph and coloured pseudocomplex are strictly related; in fact, any $(n+1)$-coloured graph (Γ, γ) may be thought of as the dual 1-skeleton of a coloured n-pseudocomplex $K = K(\Gamma)$ (whose n-simplices are in bijection with the vertices of Γ), so that an edge-coloration γ is naturally induced by that of K (i.e.: for each $e \in E(\Gamma)$, $\gamma(e) = i$ iff the vertices of the $(n-1)$-simplex of K dual to e are coloured by $\Delta_n \setminus \{i\}$).

For details about both constructions from edge-coloured graphs to coloured pseudocomplexes and viceversa, we refer to [19] and [3].

If polyhedron $|K(\Gamma)|$ is PL-homeomorphic to an n-manifold M^n, then (Γ, γ) is called a *gem* (graph encoded manifold) of M^n, or an edge-coloured graph *representing* M^n, while $K = K(\Gamma)$ is said to be a *coloured triangulation* of M^n. Furthermore, if (Γ, γ) is contracted, i.e. $\Gamma_{\hat{i}}$ is connected, for each $i \in \Delta_n$ (equivalently: $K = K(\Gamma)$ contains only one i-coloured vertex, for each $i \in \Delta_n$), it is called a *crystallization* of M^n.

It is easy to see that M^n is orientable iff any edge-coloured graph (Γ, γ) representing it is bipartite.

Classical results (see [19]) assure that each n-manifold admits a crystallization; obviously, it generally admits many and it is a basic problem how to recognize crystallizations (or, more generally, gems) of the same manifold.

The easiest case is that of two *colour-isomorphic* gems, i.e. if there exists an isomorphism between the graphs, which preserves colours up to a permutation of Δ_n. It is quite trivial to check that two colour-isomorphic gems produce the same polyhedron.

The following result assures that colour-isomorphic graphs can be effectively detected by means of a suitably defined numerical *code*, which can be directly computed on each of them (see [3]).

Proposition 1 *Two gems are colour-isomorphic iff their codes coincide.*

Also the problem of recognizing non-colour-isomorphic gems representing the same manifold is solved, but not algorithmically: a finite set of moves - the so called *dipole moves* - is proved to exist, with the property that two gems represent the same manifold iff they can be related by a finite sequence of such moves ([18]).

However, in this paper, whose results concern dimension three, we will use another set of moves - *generalized dipole moves* - defined only for 4-coloured graphs (see section 3).

Even if they still do not solve algorithmically the problem for general 3-manifolds, nevertheless we will prove that a fixed sequence of generalized dipole moves is sufficient

[4] All notations of general graph theory are given in accordance to [38].

for classifying all *rigid* crystallizations of 3-manifolds having at most 30 vertices (see sections 3 and 4).

The definition of *rigid* crystallization requires some preliminaries.

Definition 2. A pair (e, f) of distinct i-coloured edges in a 4-coloured graph (Γ, γ) is said to form a ρ_m*-pair* $(m = 2, 3)$ iff e and f share exactly m bicoloured cycles of Γ.

By a ρ*-pair* we mean a ρ_m-pair, for $m \in \{2, 3\}$.

Definition 3. A crystallization (Γ, γ) of a 3-manifold M^3 is called *rigid* if it contains no ρ-pairs.

The restriction to the class of rigid crystallizations doesn't affect the set of represented 3-manifolds, as the following result proves:

Proposition 2 ([11]) *Every closed connected 3-manifold M^3 admits a rigid crystallization. Moreover, if M^3 is handle-free*[5] *and (Γ, γ) is any gem of M^3, with $\#V(\Gamma) = 2p$, then there exists a rigid cristallization $(\bar{\Gamma}, \bar{\gamma})$ of M^3, with $\#V(\bar{\Gamma}) \leq 2p$.*

Hence, in order to obtain a list of all handle-free 3-manifolds represented by edge-coloured graphs with at most $2p$ vertices (i.e., according to [25] and [11], with *gem-complexity* $\leq p - 1$), it is sufficient to take into account rigid crystallizations only.

Remark 1 If (e, f) is a ρ_3-pair in a gem (Γ, γ) representing a 3-manifold M^3 and (Γ', γ') is the gem obtained from (Γ, γ) by *switching the ρ-pair* (see Figure 1, or [25] for details), then (Γ', γ') represents a 3-manifold J so that $M^3 = J\#H$, H being either the orientable or non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$.

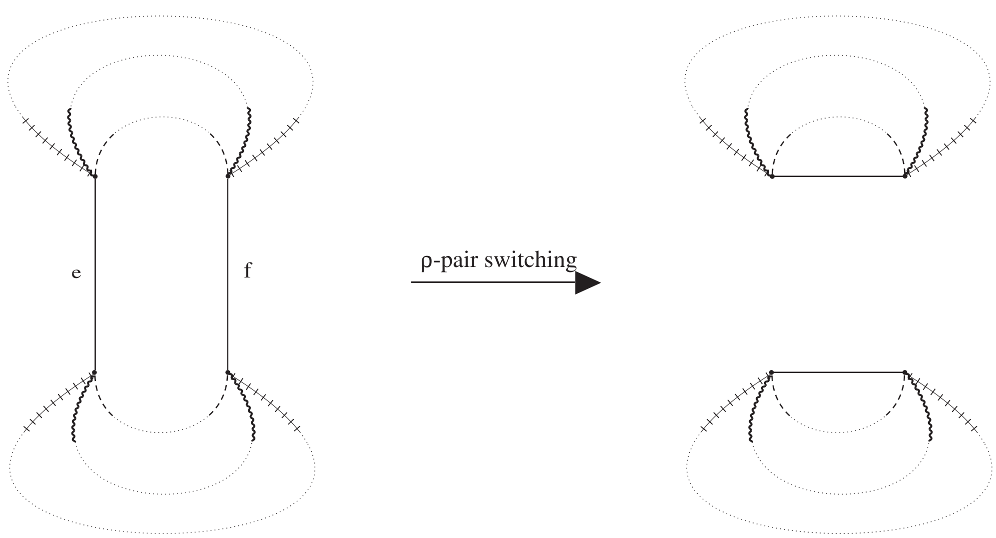

Figure 1

[5] M^3 contains a handle if it admits a decomposition $M^3 = J\#H$, where H denotes either the orientable or non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ and J is a suitable non-empty 3-manifold (possibly homeomorphic to $\mathbb{S}^3$).

Finally, let us point out that an $(n+1)$-coloured graph (Γ,γ) is a crystallization of an n-manifold iff $\Gamma_{\hat{i}}$ is a gem of $\mathbb{S}^{n-1}$, for each $i \in \Delta_n$ ([19]).

Since a 3-coloured graph represents $\mathbb{S}^2$ iff it is planar[6], the characterization of crystallizations (Γ,γ) for $n=3$ requires $\Gamma_{\hat{i}}$ to be planar and connected for each $i \in \Delta_n$.

3. Automatic cataloguing and classifying closed 3-manifolds

Combinatorial encoding of closed 3-manifolds by crystallizations allows us to construct an essential catalogue of all contracted triangulations of closed 3-manifolds up to a certain number of vertices. Moreover, Proposition 2 tells us that we can restrict our attention to rigid crystallizations; this fact yields a basic improvement in the direction of the concrete realization of the catalogue. By using the codes, we can easily avoid isomorphic graphs, too.

For every $p \in \mathbb{N}$, let $\mathcal{C}^{(2p)}$ (resp. $\tilde{\mathcal{C}}^{(2p)}$) be the catalogue of all non-isomorphic rigid bipartite (resp. non-bipartite) crystallizations with $2p$ vertices.

The generating algorithm for $\mathcal{C}^{(2p)}$ and $\tilde{\mathcal{C}}^{(2p)}$ was originally described in [11] and it consists of the following steps.

Step 1: We construct the set $\mathcal{S}^{(2p)} = \{\Sigma_1^{(2p)}, \Sigma_2^{(2p)}, \dots, \Sigma_{n_p}^{(2p)}\}$ of all (connected) rigid and planar 3-coloured graphs with $2p$ vertices. The construction makes use of the results of [24] and [25] and is performed by induction on p.

Step 2: For each $i = 1, 2, \dots, n_p$, we add to $\Sigma_i^{(2p)}$ 3-coloured edges in all possible ways to produce 4-coloured graphs, provided that:

- no vertices belonging to the same bicoloured cycle are joined (in particular no multiple edges are created [7]) to satisfy the rigidity condition.
- for each $m \in \{1, \dots, p\}$, supposing ${}^m\Lambda$ to be a 4-coloured graph (with boundary) obtained from $\Sigma_i^{(2p)}$ by adding $m < p$ 3-coloured edges, the subgraphs ${}^m\Lambda_{\hat{r}}$, for every $r \in \{0,1,2\}$, are planar. This planarity condition can be easily checked since

$$ {}^m\Lambda_{\hat{r}} \text{ is planar} \quad \text{iff} \quad 2g_{\hat{r}} - {}^{\partial}g_{\hat{r}} = \sum_{i,j \in \Delta_3 - \{r\}} \dot{g}_{ij} - m $$

where $2g_{\hat{r}}$ (resp. ${}^{\partial}g_{\hat{r}}$) is the number of connected components (resp. of not regular connected components) of ${}^m\Lambda_{\hat{r}}$ and $\dot{g}_{ij}$ is the number of closed $\{i,j\}$-coloured cycles of ${}^m\Lambda$.

[6] A 3-coloured graph is *planar* iff it has a cellular embedding in $\mathbb{R}^2$, whose 2-cells are bounded by images of bicoloured cycles.

[7] The only case where multiple edges are allowed is $p=1$: the order two 3-coloured graph consisting of three multiple edges is rigid and planar, and obviously gives rise to a rigid order two crystallization of $\mathbb{S}^3$ by addition of another (3-coloured) multiple edge.

- the resulting 4-coloured (regular) graphs $\{\Gamma^{(2p)}_{i,1}, \Gamma^{(2p)}_{i,2}, \ldots, \Gamma^{(2p)}_{i,m_i}\}$ are crystallizations (i.e. $\chi(K(\Gamma^{(2p)}_{i,j})) = 0$, for each $j \in \{1, \ldots, m_i\}$).

Step 3: Let $Y^{(2p)} = \{\Gamma^{(2p)}_{i,j}\}_{i=1,\ldots,n_p \quad j=1,\ldots,m_i}$ be the set of crystallizations arising from Steps 1 and 2. Then, by computing and comparing the codes and by checking the bipartition property, we construct the set $X^{(2p)}$ (resp. $\tilde{X}^{(2p)}$) consisting of all non-isomorphic bipartite (resp. non-bipartite) elements of $Y^{(2p)}$.

Step 4: The rigidity condition is checked on the elements of $X^{(2p)}$ (resp. $\tilde{X}^{(2p)}$) and the catalogue $\mathcal{C}^{(2p)}$ (resp. $\tilde{\mathcal{C}}^{(2p)}$) is obtained. It contains all rigid bipartite (resp. non-bipartite) crystallizations with $2p$ vertices.

The above algorithm was implemented in a C++ program, whose output data are presented in Table 1 according to the number of vertices.

2p	$\#\mathcal{S}^{(2p)}$	$\#\mathcal{C}^{(2p)}$	$\#\tilde{\mathcal{C}}^{(2p)}$
2	1	1	0
4	0	0	0
6	0	0	0
8	2	1	0
10	0	0	0
12	1	1	0
14	1	1	1
16	2	3	1
18	2	4	1
20	8	23	9
22	8	44	12
24	32	262	88
26	57	1252	480
28	185	7760	2790
30	466	56912	21804

Table 1: rigid crystallizations up to 30 vertices.

Crystallizations of catalogues $\mathcal{C}^{(2p)}$ (resp. $\tilde{\mathcal{C}}^{(2p)}$) up to $p = 14$ (resp. $p = 13$) were investigated and the related manifolds identified in [25] (resp. [11]).

In this paper we face the problem of identifying the 3-manifolds represented by catalogue $\mathcal{C}^{(30)}$ (forthcoming papers will take into account the same problem for $\tilde{\mathcal{C}}^{(28)}$ and $\tilde{\mathcal{C}}^{(30)}$).

However, our procedure is completely general and, when applied to $\mathcal{C}^{(2p)}$ and $\tilde{\mathcal{C}}^{(2p)}$ for $p \leq 14$ and $p \leq 13$ respectively, has yielded the known results.

The basic tool is the possibility of subdividing a given set X of crystallizations into subsets (*classes*) such that each class contains only crystallizations representing the same manifold.

Obviously, our hope is to obtain classes large enough to coincide with the topological homeomorphism classes of the manifolds represented by the elements of X.

To fulfill this aim, we need further notions from crystallization theory.

Definition 4. Let (Γ, γ) be a gem of a closed connected 3-manifold M^3. If there exists an $\{i, j\}$-coloured cycle of length $m + 1$ and a $\{k, l\}$-coloured cycle of length $n + 1$ in Γ (with $\{i, j, k, l\} = \Delta_3$) having exactly one common vertex $\bar{v}$, then (Γ, γ) is said to contain a *(m,n)-generalized dipole* of type $\{i, j\}$ at vertex $\bar{v}$.

To *cancel* a (m, n)-generalized dipole from a gem (Γ, γ) means to perform on (Γ, γ) the operation visualized in Figure 2 (in case $m = 3; n = 5$).

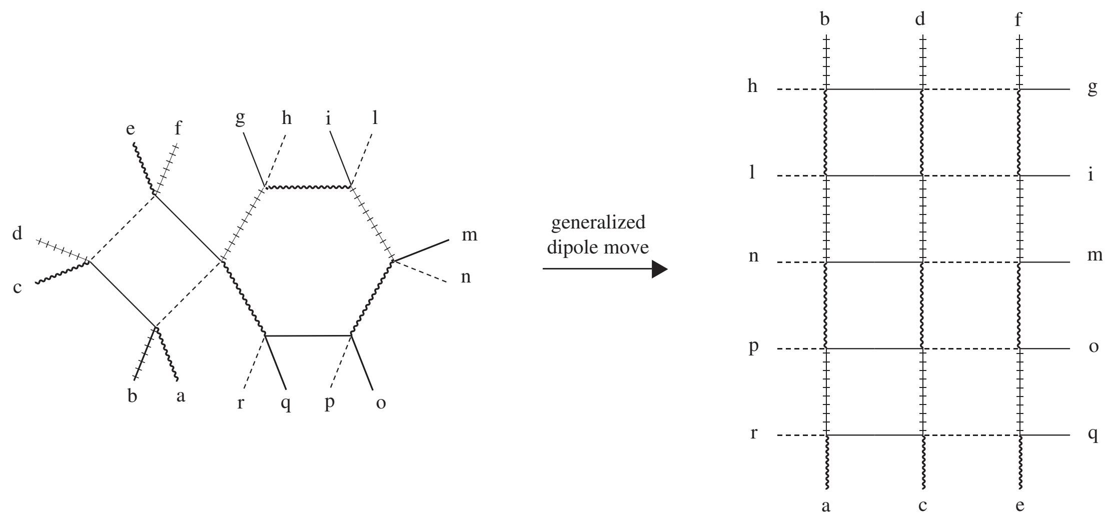

Figure 2

In the following we refer to the cancellation of a (m, n)-generalized dipole and to its inverse procedure as *generalized dipoles moves*.

It is a known result (see [18]) that every two gems which are transformed into each other by a sequence of generalized dipoles moves represent the same manifold.

Therefore generalized dipoles moves are a useful tool to manipulate crystallizations without changing the represented manifolds.

Let (Γ, γ) be a rigid crystallization and suppose that in $V(\Gamma)$ an ordering is fixed; given an integer $i \in \{1, 2, 3\}$, we denote by $\theta_i(\Gamma)$ the rigid crystallization obtained from (Γ, γ) by subsequent cancellations of (m, n)-dipoles of type $\{0, i\}$, according to the following rules:

- $m, n < 9$ (this condition is necessary to bound the possible number of vertices of $\theta_i(\Gamma)$).[8]

- supposing $V(\Gamma) = \{v_1, \ldots, v_{2p}\}$, with vertex labelling coherent with the fixed ordering, the generalized dipoles of type $\{0, i\}$ are looked for and cancelled for increasing value of the integer $m \cdot n$ and by starting from vertex v_1 up to v_{2p};

[8] Cancellation of a generalized dipole increases the number of vertices, but dipoles are frequently created as a consequence, and their further cancellation allows to decrease the number of vertices.

this means that, if $\delta(v_i)$ (resp. $\delta'(v_j)$) is a $(m,n)-$ (resp. a $(m',n')-$) generalized dipole at vertex v_i (resp. v_j), then the cancellation of $\delta(v_i)$ is performed before the cancellation of $\delta(v_j)$ iff $m \cdot n < m' \cdot n'$, or ($m \cdot n = m' \cdot n'$ and $i < j$).

- after each generalized dipole cancellation, proper dipoles and ρ-pairs, if any, are cancelled in the resulting graph.

Moreover, we define $\theta_0(\Gamma) = (\Gamma, \gamma)$.

Note that, given a rigid crystallization (Γ,γ), there is an obvious procedure which, starting from the code of (Γ,γ), yields a rigid crystallization $(\Gamma^<, \gamma^<)$ which is colour-isomorphic to (Γ,γ) and such that an ordering is induced in $V(\Gamma^<)$ by the *rooted numbering algorithm* generating the code (see [25]). As a consequence, for each $i \in \{0,1,2,3\}$, we can define a map θ_i on any set X of rigid crystallizations by setting, for each $(\Gamma,\gamma) \in X$, $\theta_i(\Gamma) = \theta_i(\Gamma^<)$, with the ordering of the vertices induced by the code of (Γ,γ).

Let us define the set $S_3^0 = \{\varepsilon = (\varepsilon_0 = 0, \varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon \text{ is a permutation of } \Delta_3\}$. For each $\varepsilon \in S_3^0$ and for each $i \in \Delta_3$ we set

$$\theta_{\ll \varepsilon_i \gg} = \theta_{\varepsilon_i} \circ \theta_{\varepsilon_{i-1}} \circ \ldots \circ \theta_{\varepsilon_0}$$

and denote by $h_{\ll \varepsilon_i \gg}(\Gamma)$ the number of ρ_3-pairs which has been deleted when transforming $\Gamma^<$ in $\theta_{\ll \varepsilon_i \gg}(\Gamma^<)$ (obviously if no ρ_3-pair was deleted, we set $h_{\ll \varepsilon_i \gg}(\Gamma) = 0$).

Now we are ready to describe the algorithm which, working on a given list X of rigid crystallizations, produces a partition of X into equivalence classes, $\{cl(\Gamma) \; / \; \Gamma \in X\}$, such that, $\forall \Gamma' \in cl(\Gamma)$, Γ' and Γ represent the same 3-manifold M *up to addition of handles*, i.e. there exist $h, k \in \mathbb{N} \cup \{0\}$ such that $|K(\Gamma)| = M \#_h H$ and $|K(\Gamma')| = M \#_k H$, where $H = S^1 \times S^2$ or $H = S^1 \tilde{\times} S^2$ according to the bipartition of Γ and Γ'.[9]

The basic idea is that two crystallizations $(\Gamma,\gamma), (\Gamma',\gamma')$ of X belong to the same class iff there exist $\varepsilon, \mu \in S_3^0$ and $i, j \in \Delta_3$ such that $\theta_{\ll \varepsilon_i \gg}(\Gamma)$ and $\theta_{\ll \mu_j \gg}(\Gamma')$ have the same code.

We consider X as an ordered list and we shall write $\Gamma \prec \Gamma'$ if Γ comes before Γ' in X.

For each $(\Gamma,\gamma) \in X$, the construction of $cl(\Gamma)$ is performed via the following algorithm.

Step 1: Set $cl(\Gamma) = \{\Gamma\}$ and $h(\Gamma) = 0$.

Step 2: For each $\varepsilon \in S_3^0$, $i \in \Delta_3$ and for each $\Gamma' \in X$ with $\Gamma' \prec \Gamma$, if there exist $\mu \in S_3^0$ and $j \in \Delta_3$ such that the codes of $\theta_{\ll \varepsilon_i \gg}(\Gamma)$ and $\theta_{\ll \mu_j \gg}(\Gamma')$ coincide, then

- if $h(\Gamma') - h_{\ll \mu_j \gg}(\Gamma') \geq h(\Gamma) - h_{\ll \varepsilon_i \gg}(\Gamma)$, set $h(\Gamma'') = k - h(\Gamma) + h_{\ll \varepsilon_i \gg}(\Gamma) + h(\Gamma') - h_{\ll \mu_j \gg}(\Gamma')$ for each $\Gamma'' \in cl(\Gamma)$ with $h(\Gamma'') = k$;
- if $h(\Gamma') - h_{\ll \mu_j \gg}(\Gamma') < h(\Gamma) - h_{\ll \varepsilon_i \gg}(\Gamma)$, set $h(\Gamma'') = k + h(\Gamma) - h_{\ll \varepsilon_i \gg}(\Gamma) - h(\Gamma') + h_{\ll \mu_j \gg}(\Gamma')$ for each $\Gamma'' \in cl(\Gamma')$ with $h(\Gamma'') = k$;

In both cases, set $c = cl(\Gamma) \cup cl(\Gamma')$ and $cl(\Gamma'') = c$ for each $\Gamma'' \in c$.

[9] More precisely, $H = S^1 \times S^2$ iff Γ and Γ' are both bipartite or both non-bipartite.

Furthermore, for each class $c_i = \{\Gamma^i_1, \dots, \Gamma^i_{r_i}\}$ and for each $0 \le h \le max\{h(\Gamma^i_1), \dots, h(\Gamma^i_{r_i})\}$, we define a partition of c_i into subsets $c_{i,h} = \{\Gamma^i_j \in c_i \mid h(\Gamma^i_j) = h\}$.

Via Proposition 2 and Remark 1 it is very easy to check that, if $\Gamma \in X$ represents the manifold M with $h(\Gamma) = h$ and $c_i = cl(\Gamma)$, then each element of $c_{i,k}$ $(0 \le k \le max\{h(\Gamma') \mid \Gamma' \in c_i\})$ represents the manifold M' with $M' = M\#_{k-h}H$ or $M = M'\#_{h-k}H$ (where $H = S^1 \times S^2$ or $H = S^1 \tilde{\times} S^2$, as above), according to $k \ge h$ or $k < h$.

Remark 2 Note that the algorithm works for any chosen sequence of generalized dipoles moves; to improve the results of our implementation, for example, we choose to compare not only the graphs $\theta_{\ll\varepsilon_i\gg}(\Gamma)$ but also the graphs

$$\theta_{\ll\varepsilon_i^{(k)}\gg} \circ \theta_{\ll\varepsilon_3^{(k-1)}\gg} \circ \dots \circ \theta_{\ll\varepsilon_3^{(1)}\gg}(\Gamma),$$

where $\varepsilon^{(k)}$ $(k \in \{2, \dots, 6\})$ is the k-th permutation of S^0_3 considered as a lexicographically ordered set.

In order to analyze and - possibly - identify topologically the manifolds represented by the crystallizations of a given list X, the first step consists in looking for crystallizations in X which are already identified (as it happens, for example, if known catalogues of crystallizations are contained in X). In this case, if the information is added to the input data of our algorithm, all classes c_i containing at least one of the known crystallizations, together with their possible subclasses $c_{i,h}$, turn out to be completely identified: their elements represent $M\#_t(S^1 \times S^2)$ for a fixed manifold M and for a convenient value of the handle number t (according to the previously described rules).

A second step in the direction of the topological identification of the given list X is to try connected sums recognition among manifolds represented by unknown classes. In order to write explicitly the involved combinatorial condition, we need further results from crystallization theory.

Proposition 3 *Let (Γ, γ) be a 3-gem representing a closed connected 3-manifold M. Suppose there exist four edges $\{e_0, e_1, e_2, e_3\}$ in (Γ, γ) such that $\gamma(e_i) = i$, for each $i \in \Delta_3$, and $\Gamma \setminus \{e_0, e_1, e_2, e_3\}$ has two connected components. Then, there exist 3-gems $(\Gamma^{(1)}, \gamma^{(1)})$ and $(\Gamma^{(2)}, \gamma^{(2)})$ such that $M = |K(\Gamma^{(1)})|\#|K(\Gamma^{(2)})|$.*

Proof. Let $(\bar{\Gamma}^{(i)}, \bar{\gamma}^{(i)})$, for $i = 1, 2$, be the connected components of $\Gamma \setminus \{e_0, e_1, e_2, e_3\}$. For each $i = 1, 2$, let us consider the 4-coloured (regular) graph $(\Gamma^{(i)}, \gamma^{(i)})$ obtained from $(\bar{\Gamma}^{(i)}, \bar{\gamma}^{(i)})$ by adding a new vertex v_i and four edges $\{e_{i0}, e_{i1}, e_{i2}, e_{i3}\}$ such that $\gamma^{(i)}(e_{ij}) = j$ $(j \in \Delta_3)$ and e_{ij} is incident to v_i and the (boundary) vertex of $(\bar{\Gamma}^{(i)}, \bar{\gamma}^{(i)})$ missing colour j. It is easy to see that $(\Gamma^{(i)}, \gamma^{(i)})$ is a 3-gem and (Γ, γ) is the *connected sum* of $(\Gamma^{(1)}, \gamma^{(1)})$ and $(\Gamma^{(2)}, \gamma^{(2)})$ with respect to the vertices v_1 and v_2 (see [19], or Figure 3 for an example). Therefore M is the connected sum of $|K(\Gamma^{(1)}|$ and $|K(\Gamma^{(2)})|$ (see [19]). □

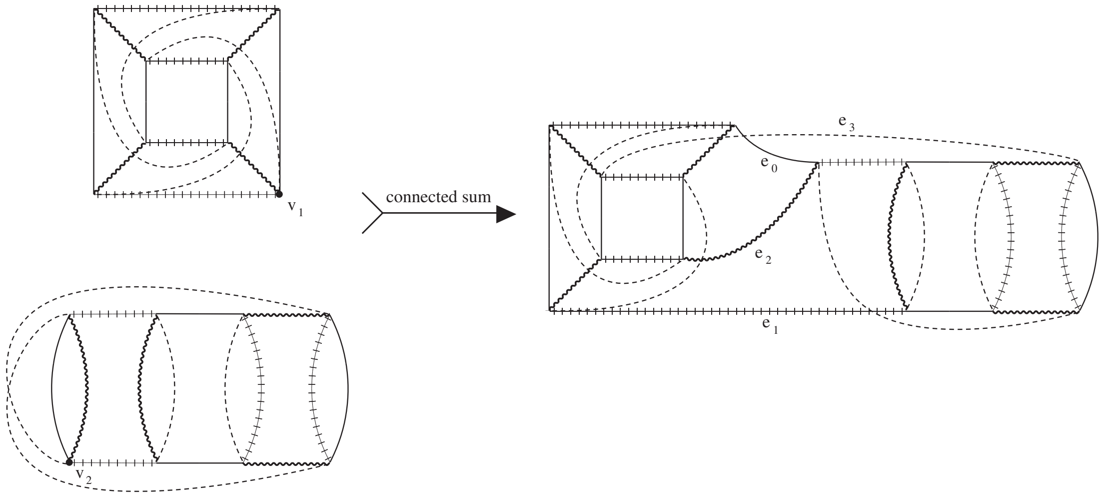

Figure 3

Proof of Proposition 3 tells us that whenever a crystallization satisfies the condition of the statement we can split it and analyze the resulting "pieces".

Remark 3 Of course, it is possible that one of the gems $(\Gamma^{(i)}, \gamma^{(i)})$ represents S^3, i.e. M splits in a trivial connected sum; in any case, the two gems have fewer vertices than (Γ, γ) and they will probably be easier to be recognized.

The C++ program implementing our "classification" algorithm was applied to the catalogue $\mathbf{C}^{(30)} = \bigcup_{1 \leq p \leq 15} \mathcal{C}^{(2p)}$; it produced 172 classes, 100 of which completely recognized by means of the existing results about catalogues $\mathcal{C}^{(2p)}$, for $1 \leq p \leq 14$.

Another C++ program was used to check the condition of Proposition 3 on the crystallizations of every class; whenever the condition was satisfied by a representative (Γ, γ) of a class, the crystallizations $(\Gamma^{(i)}, \gamma^{(i)})$ $(i = 1, 2)$ of Proposition 3 were constructed and compared, by the code, with the already known crystallizations of the catalogues $\mathcal{C}^{(2p)}$, for $1 \leq p \leq 14$.

Sixty-two classes where thus recognized; thirty-one of these connected sums already appeared in catalogues $\mathcal{C}^{(2p)}$, for $1 \leq p \leq 14$.

In the following section we will present the analysis of the above data, which allowed us to obtain the results of Theorem I and the complete topological classification of the manifolds encoded by the crystallizations of catalogue $\mathbf{C}^{(30)}$.

4. A complete analysis of catalogue $\mathbf{C}^{(30)}$

Before discussing our experimental results on $\mathbf{C}^{(30)}$, it is useful to recall the already known topological identification of the manifolds involved in $\mathcal{C}^{(2p)}$, for $1 \leq p \leq 14$.

The classification, which is based on the results of [25], was proved in [15] and used for studying a combinatorial concept of complexity and its relation with Matveev's complexity; the following statement reproduces it in a slightly different form as to suit our present aims.

Proposition 4 *There exist exactly sixty-nine closed connected prime orientable 3-manifolds, which admit a coloured triangulation consisting of at most 28 tetrahedra.*

Among them, there are:

- $\mathbb{S}^3$*;*
- $\mathbb{S}^2 \times \mathbb{S}^1$*;*
- *the six Euclidean orientable 3-manifolds;*
- *forty-four elliptic 3-manifolds (in particular, twenty-three lens spaces);*
- *fifteen Seifert non-elliptic 3-manifolds*

  (more precisely:

  - *four torus bundles with Nil geometry;*
  - *three manifolds of type* $(K \tilde{\times} I) \cup (K \tilde{\times} I)/A \quad (A \in GL(2;\mathbb{Z}), \ \det(A) = -1)$, *with Nil geometry;*
  - *seven manifolds with* $\widetilde{SL_2}(\mathbb{R})$ *geometry;*
  - *a further manifold with* Nil *geometry);*
- *two torus bundles with Sol geometry.*

The first step towards the identification of the crystallizations of $\mathbf{C}^{(30)}$ was to compare the results of the classification program with the known catalogues $\mathcal{C}^{(2p)}$ with $1 \leq p \leq 14$. This was made by the classifying program itself.

More precisely, whenever the algorithm produced a crystallization having less than 30 vertices, the program searched for it in the known catalogues and gave the resulting name to its class.

In this way 100 classes were recognized; this is exactly the number of 3-manifolds admitting a rigid crystallization with $2p$ vertices, $1 \leq p \leq 14$.

Actually, a careful examination of our output data allows to state the following:

Lemma 5 *The set of classes obtained from* $\mathbf{C}^{(30)}$ *and containing at least one crystallization with less than 30 vertices is in bijective correspondence with the set of 3-manifolds represented by* $\mathcal{C}^{(2p)}$, $\ 1 \leq p \leq 14$.

Further identifications were obtained by the analysis of the output of the "connected sum" program, as described in the above section.

Let us introduce a notation, which will be useful in the following.

Let M, N be two closed orientable 3-manifolds; we denote by $\mathcal{C}(M,N)$ the set of classes $cl(\Gamma)$ of crystallizations $(\Gamma,\gamma) \in \mathbf{C}^{(30)}$ such that (Γ,γ) satisfies the condition of Proposition 3, with summands $(\Gamma^{(1)},\gamma^{(1)})$, $(\Gamma^{(2)},\gamma^{(2)})$ and $\{|K(\Gamma^{(1)})|, |K(\Gamma^{(2)})|\} = \{M, N\}$.

We can summarize our results by the following statement.

Lemma 6

(i) For each $(\Gamma,\gamma) \in \mathbf{C}^{(30)}$ satisfying the condition of Proposition 3 with summands $(\Gamma^{(1)},\gamma^{(1)})$ and $(\Gamma^{(2)},\gamma^{(2)})$, then $(\Gamma^{(i)},\gamma^{(i)})$ belongs to $\mathbf{C}^{(28)} = \bigcup_{1\le p\le 14} \mathcal{C}^{(2p)}$, for each $i = 1,2$;

(ii) given two closed orientable 3-manifolds M, N such that $\mathcal{C}(M,N) \neq \emptyset$, we have

- *$\#\mathcal{C}(M,N) = 1$ iff at least one of M, N admits orientation-reversing self-homeomorphisms;*
- *$\#\mathcal{C}(M,N) = 2$ iff neither M nor N admit orientation-reversing self-homeomorphisms;*

(iii) If $c, c' \in \mathcal{C}(M,N)$, with $c \neq c'$, then c and c' represent non-homeomorphic manifolds.

Proof. Statements (i) and (ii) have been deduced directly by the classification program results. With regard to statement (iii) more details are required. Let us consider two classes c and c' as specified above. We have proved that they represent different manifolds by the following steps:

- In virtue of statement (ii), the hypothesis of statement (iii) implies $\mathcal{C}(M,N) = \{c, c'\}$, with both M and N not admitting orientation-reversing self-homeomorphisms; moreover, there exists a representative of c (resp. c'), which is a connected sum of a crystallization of M and a crystallization of N;
- let (Γ_1,γ_1) (resp. (Γ_2,γ_2)) be the first crystallization of catalogue $\mathbf{C}^{(28)}$ representing M (resp. N);
- for each $i = 1,2$ fix a bipartition on the set $V(\Gamma_i)$, choose a vertex $v_i \in V(\Gamma_i)$ $(i = 1,2)$ and perform the connected sum of graphs (Γ_1,γ_1) and (Γ_2,γ_2) with respect to vertices v_1 and v_2, denoting by $\Gamma_1^+\#\Gamma_2^+$ the resulting crystallization;
- construct the connected sum of (Γ_1,γ_1) and (Γ_2,γ_2) with respect to v_1 and a vertex $w \in V(\Gamma_2)$ such that v_2 and w belong to different bipartition classes of $V(\Gamma_2)$, and denote it by $\Gamma_1^+\#\Gamma_2^-$;

It is easy to see that $\Gamma_1^+\#\Gamma_2^+$ and $\Gamma_1^+\#\Gamma_2^-$ represent non-homeomorphic manifolds (see [21]), which will be denoted by $M^+\#N^+$ and $M^+\#N^-$.

By applying the classification program to the list formed by $\Gamma_1^+\#\Gamma_2^+$, $\Gamma_1^+\#\Gamma_2^-$ and the crystallizations of c and c', we have obtained that there are exactly two classes $\bar{c}$ and $\bar{c}'$ such that $\bar{c}$ (resp. $\bar{c}'$) contains all crystallizations in c (resp. c') and one element of $\{\Gamma_1^+\#\Gamma_2^+, \Gamma_1^+\#\Gamma_2^-\}$ (resp. the other element of $\{\Gamma_1^+\#\Gamma_2^+, \Gamma_1^+\#\Gamma_2^-\}$), i.e. c and c' actually

represent the distinct manifolds $M^+\#N^+$ and $M^+\#N^-$.

$\square$

Remark 4 With regard to the manifolds M, N such that $\#\mathcal{C}(M,N) = 2$, we point out that in all cases except one the manifolds involved are $L(3,1)$ and either a lens space $L(p,q)$ with $(p,q) \in \{(5,1),(7,2),(8,3)\}$ or the elliptic manifold S^3/Q_{12}. The remaining case is the sum of two copies of $L(4,1)$.

After the comparison with known catalogues $\mathcal{C}^{(2p)}$, $1 \leq p \leq 14$, and splitting as connected sum, exactly forty-one unknown classes of crystallizations in $\mathbf{C}^{(30)}$ turned out to be still unrecognized.

In order to complete the topological identification of all represented 3-manifolds, a representative for each unknown class was handled by *Three-manifold Recognizer*[10], the program written by V. Tarkaev as an application of the results about recognition of 3-manifolds obtained by S.Matveev and his research group.

The output of Matveev-Tarkaev's program proves that all forty-one classes under examination are topologically distinct and represent prime manifolds; therefore, by making use also of our former analysis (see Lemma 5 and Lemma 6), we can state the following:

Proposition 7

(i) *There is a bijective correspondence between the set of equivalence classes obtained by the classification program and the set of 3-manifolds represented by* $\mathbf{C}^{(30)}$*;*

(ii) *all connected sums in* $\mathbf{C}^{(30)}$ *are identified by the connected sum program, i.e. each class representing a connected sum contains at least one crystallization satisfying Proposition 3.*

With regard to prime 3-manifolds not appearing in $\mathbf{C}^{(28)}$, the above described analysis of catalogue $\mathcal{C}^{(30)}$ may be summarized by the following statement, which directly implies (via Proposition 2) Theorem I.

Proposition 8 *There exist exactly forty-one closed connected prime orientable 3-manifolds, whose minimal coloured triangulation consists of 30 tetrahedra.*
Among them, there are:

- *10 elliptic 3-manifolds;*
- *17 Seifert non-elliptic 3-manifolds (in particular, 2 torus bundles with Nil geometry);*
- *2 torus bundles with Sol geometry;*
- *2 manifolds of type* $(K \tilde{\times} I) \cup (K \tilde{\times} I)/A \quad (A \in GL(2;\mathbb{Z}),\ \det(A) = -1)$*, with Sol geometry;*
- *7 non-geometric graph manifolds;*

[10]It is available on the Web: http://www.topology.kb.csu.ru/∼recognizer/

- *3 hyperbolic Dehn-fillings (of the complement of link 6_1^3).*

Remark 5 Some of the "new" prime 3-manifolds represented by elements of $\mathcal{C}^{(30)}$ can be actually identified also within crystallization theory, without the aid of Matveev-Tarkaev's program:

- the four torus bundles $TB(A)$ (i.e. those obtained with $A \in \{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}\}$) have also been recognized by direct construction of the corresponding edge-coloured graphs $\Gamma_{TB}(A)$ (see [12]), and then by applying, for each A, the classification program to the list formed by $\Gamma_{TB}(A)$ and the crystallizations of the only class admitting the group $\pi_1(TB(A))$ as fundamental group.

- the two manifolds of type $KB(A) = (K \tilde{\times} I) \cup (K \tilde{\times} I)/A$ (i.e. those obtained with $A \in \left\{\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\right\}$) have also been recognized by direct construction of the corresponding edge-coloured graphs $\Gamma_{KB}(A)$ (see [14]), and then by applying, for each A, the classification program to the list formed by $\Gamma_{KB}(A)$ and the crystallizations of the only class admitting the group $\pi_1(KB(A))$ as fundamental group.

Moreover:

- all 10 elliptic 3-manifolds (i.e.: S^3/D_{80}, S^3/D_{112}, $S^3/(Q_{28} \times Z_5)$, $S^3/(Q_{32} \times Z_5)$, $S^3/(P_{48} \times Z_{11})$, $S^3/(P_{48} \times Z_5)$, $S^3/(P_{48} \times Z_7)$, $S^3/(P_{120} \times Z_{23})$, $S^3/(P_{120} \times Z_{17})$, $S^3/(P_{120} \times Z_{13})$) and the Seifert manifold $SFS(S^2, (2,1), (4,1), (5,2), (1,-1))$, with SL_2R geometry, have also been recognized directly by means of homology computation and/or analysis of a presentation of the fundamental group[11], together with an estimation of the complexity via GM-complexity (see [13]);

- two further manifolds (i.e. the Seifert manifolds $SFS((3,1), (3,2), (3,2), (1,-1))$, with Nil geometry, and $SFS(K, (2,1))$), with SL_2R geometry) are easily recognized by homology computation, together with an estimation of the complexity via GM-complexity (see [13]).

The complete list of the forty-one closed connected prime orientable 3-manifolds represented by $\mathcal{C}^{(30)}$ may be found in Table 2 of [16]. In analogy to the similar Table 1 of [16], containing the sixty-nine closed connected prime orientable 3-manifolds represented by $\mathbf{C}^{(28)}$ (see Proposition 4 and [15]), each manifold is identified by means of its JSJ decomposition and fibering structure, according to Matveev's description in [33] (see also [30], [31] and [32]); to make comparison easier, the position of each manifold within Matveev's table [33] is also given.

All manifolds involved in $\mathbf{C}^{(30)}$ have also been detected within Martelli-Petronio censuses of closed irreducible orientable 3-manifolds up to complexity 10 (see [29]), and

[11] In some cases, GAP program (see [20]) has been useful to handle group presentations, by computation of the corresponding order and/or analysis of low-index subgroups.

interesting results followed from a comparative analysis of both complexity and geometric properties of manifolds represented by subsequent catalogues $\mathcal{C}^{(2p)}$, $1 \leq p \leq 15$. In fact, the presence of manifolds in $\mathbf{C}^{(30)}$ with respect to complexity and geometry (see [28] for cases up to complexity 9, and [26] for the last case) may be summarized in the following table, where the symbol x/n means that x 3-manifolds appear in $\mathbf{C}^{(30)}$, among the n ones having the appropriate complexity and geometry, and bold character is used to indicate that <u>all</u> manifolds of the considered type appear in catalogue $\mathbf{C}^{(30)}$:

complexity	1	2	3	4	5	6	7	8	9	10
lens	**2/2**	**3/3**	**6/6**	**10/10**	0/20	0/36	0/72	0/136	0/272	0/528
other elliptic	-	**1/1**	**1/1**	**4/4**	**11/11**	14/25	0/45	0/78	0/142	0/270
euclidean	-	-	-	-	-	**6/6**	-	-	-	-
Nil	-	-	-	-	-	**7/7**	3/10	0/14	0/15	0/15
$H^2 \times S^1$	-	-	-	-	-	-	-	0/2	-	0/8
SL_2R	-	-	-	-	-	-	13/39	0/162	0/513	0/1416
Sol	-	-	-	-	-	-	4/5	2/9	0/23	0/39
non-geometric	-	-	-	-	-	-	**4/4**	1/35	2/185	0/777
hyperbolic	-	-	-	-	-	-	-	-	2/4	1/25
TOTAL	**2/2**	**4/4**	**7/7**	**14/14**	11/31	27/74	24/175	3/436	4/1154	1/3078

Table 2: 3-manifolds involved in $\mathbf{C}^{(30)}$

We think worth noting that, for any fixed complexity c, catalogues $\mathcal{C}^{(2p)}$ cover, for increasing p, first the most "complicated" types of complexity c 3-manifolds and then the simplest ones: for a detailed analysis on the subject, see Table 3 of [16].[12]

As a consequence, catalogues $\mathcal{C}^{(2p)}$, for increasing value of p, appear to be a useful source for interesting examples to test conjectures and search for patterns about 3-manifolds.

5. Automatic recognition of orientable 3-manifolds

In this section we want to point out that our approach to the study of 3-manifolds with low gem-complexity yields not only the "list" of involved 3-manifolds, but also the "list" of all possible coloured triangulations of manifolds with a given number of tetrahedra.[13]

In fact, the production and analysis of catalogue $\mathcal{C}^{(30)}$, with the topological recognition of all represented 3-manifolds (see Theorem I), enables to answer positively the following general questions, which are "dual" to each other.

[12] For example note that, as far as complexity 4 (resp. 5) is concerned, all 10 lens spaces appear in $\mathcal{C}^{(28)}$, while all 4 elliptic 3-manifolds appear in $\mathcal{C}^{(22)} \cup \mathcal{C}^{(24)}$ (resp. none of the 20 lens spaces appear in $\mathbf{C}^{(30)}$, while all 11 elliptic 3-manifolds appear in $\mathcal{C}^{(24)} \cup \mathcal{C}^{(26)} \cup \mathcal{C}^{(28)}$).

[13] A similar point of view may be found in Burton's works (see [6], [7], [8], [9]) where all minimal triangulations of manifolds with low complexity are directly constructed.

Given a 4-coloured graph with $2p \leq 30$ vertices,
is it possible to say whether it represents an orientable 3-manifold M^3
and - in the affirmative - to recognize M^3?

Given a coloured 3-dimensional pseudocomplex with $2p \leq 30$ tetrahedra,
is it possible to say whether it represents an orientable 3-manifold M^3
and - in the affirmative - to recognize M^3?

A suitable option of DUKE III program [14] answers completely the above questions, for any 4-coloured graph (Γ, γ) (resp. any coloured triangulation $K = K(\Gamma)$): if either the associated matrix $A(\Gamma)$ or the code $c(\Gamma)$ of (Γ, γ) is given as input data, then - in case $|K(\Gamma)|$ is a 3-manifold M^3 - the program identifies M^3 within Matveev's catalogue of closed irreducible orientable 3-manifolds represented by special spines up to complexity 11 (see [33]), according to Table 1 and Table 2 of [16].

Proposition 9 *Let (Γ, γ) be any bipartite 4-coloured graph such that*

$$\sum_{i,j \in \Delta_3} g_{ij} - \sum_{i \in \Delta_3} g_{\hat{i}} = p \qquad \text{and} \qquad 2p - \sum_{i \in \Delta_3} (g_{\hat{i}} - 1) \leq 30,$$

where $g_{\hat{i}}$ is the number of connected components of $\Gamma_{\hat{i}}$ and g_{ij} is the number of $\{i,j\}$-coloured cycles of Γ.
Then, DUKE III program yields, by means of obtained results about catalogue $\boldsymbol{C}^{(30)}$, the topological recognition of the closed orientable 3-manifold $M^3 = |K(\Gamma)|$.

Proof. First of all, note that condition $\sum_{i,j \in \Delta_3} g_{ij} - \sum_{i \in \Delta_3} g_{\hat{i}} = p$ ensures the polyhedron $|K(\Gamma)|$ to be a PL 3-manifold M^3. Moreover, it is very easy to check that exactly $\sum_{i \in \Delta_3}(g_{\hat{i}} - 1)$ subsequent 1-dipole eliminations (each decreasing by two the number of vertices) may be performed in (Γ, γ), giving rise to a crystallization of M^3. Hence, the statement follows directly from the topological identification of the manifold represented by each element of catalogue $\mathbf{C}^{(30)}$ (see Theorem I and Proposition 4).

□

6. A more essential catalogue for handle-free 3-manifolds

As already pointed out in paragraph 3, the basic result for the concrete realization of catalogues $\mathcal{C}^{(2p)}$ and $\tilde{\mathcal{C}}^{(2p)}$ is Proposition 2, which allows to restrict the generation process to *rigid* crystallizations, without loss of generality as far as represented 3-manifolds are concerned.

In this paragraph a further, significant improvement in the same direction is presented; it relies on an idea originally due to Lins (see [25], paragraph 4.1.4, where basic

[14] See http://cdm.unimo.it/home/matematica/casali.mariarita/DukeIII.htm

concepts for the following definitions and result appear, without successive application to catalogue generation process yet).

Definition 5. Let (Γ, γ) be a gem of the 3-manifold M^3. A vertex $v \in V(\Gamma)$ is said to be a *cluster-type vertex* if (Γ, γ) has four bicoloured cycles containing v with length four, which involve exactly nine vertices of (Γ, γ).

Definition 6. A four-coloured graph (Γ, γ) is said to be a *cluster-less gem* of a 3-manifold M^3 if $|K(\Gamma)| = M^3$ and (Γ, γ) admits no cluster-type vertices.

Proposition 10 *Let (Γ, γ) be a gem of a 3-manifold M^3, with $V(\Gamma) = 2p$. If (Γ, γ) contains a cluster-type vertex, then there exists a cluster-less gem (in particular, a cluster-less crystallization) (Γ', γ') of M^3, with $\#V(\bar{\Gamma}) < 2p$. Moreover, if M^3 is handle-free and $(\bar{\Gamma}, \bar{\gamma})$ is a rigid crystallization of M^3, with $\#V(\bar{\Gamma}) = 2\bar{p}$, which contains a cluster-type vertex, then there exists a rigid cluster-less cristallization $(\bar{\Gamma}', \bar{\gamma}')$ of M^3, with $\#V(\bar{\Gamma}') < 2\bar{p}$.*

Proof.

As shown in Proposition 24 of [25] (where the hypothesis that a cluster-type vertex has to involve exactly nine vertices is actually understood), two cases may arise, for each cluster-type vertex v:

- If the bicoloured cycles with length greater than four and containing v have a common colour, then (Γ, γ) may be simplified by means of a so called TS_1-*move* (which is realized by a standard sequence of dipole moves, not affecting the number of vertices: see Figure 4, or Paragraph 4.1.2 of [25] for details about TS-moves), followed by a 2-dipole elimination.

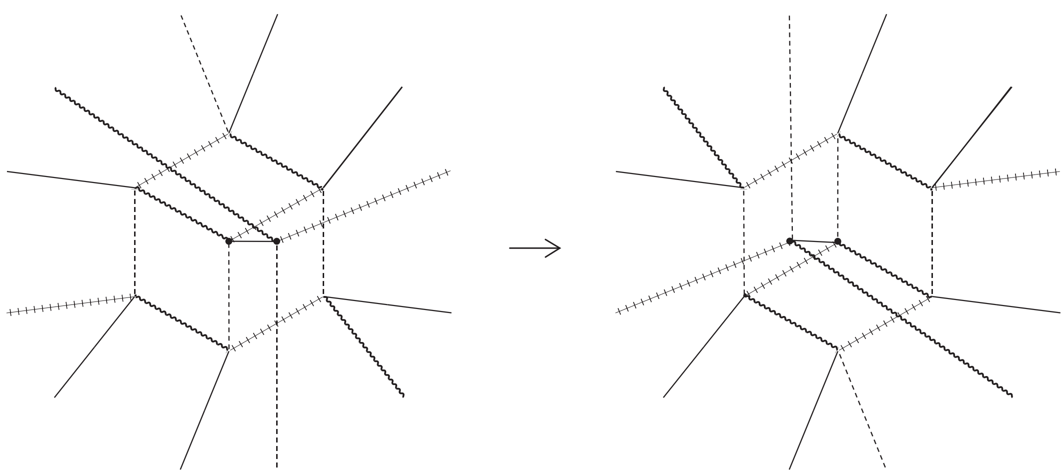

Figure 4

- If the bicoloured cycles with length greater than four and containing v have no common colours, then (Γ, γ) may be thought of as the gem obtained by elimination of a generalized dipole of type $(3, 3)$ on a suitable gem (with exactly two fewer vertices than (Γ, γ)) of the same manifold (see Figure 5).

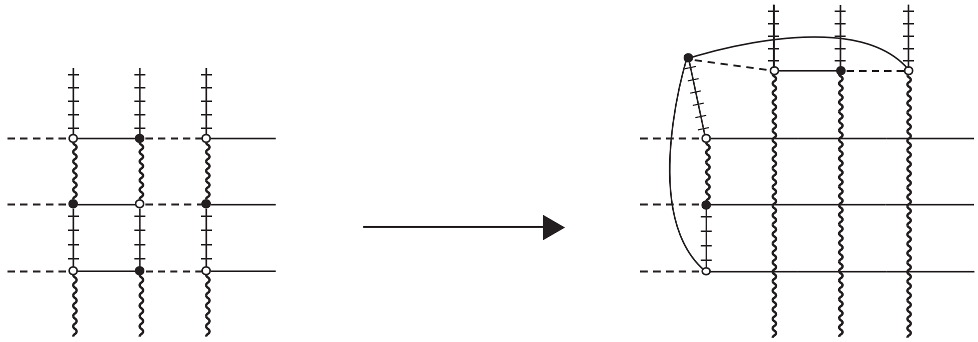

Figure 5

By iterating the process for each cluster-type vertex, a cluster-less gem (Γ', γ') of M^3 is obviously obtained, having strictly fewer vertices than (Γ, γ). Moreover, if a cluster-less crystallization of M^3 is required, it is sufficient to perform also all possible 1-dipole eliminations in (Γ', γ'), and then - if it is necessary - to repeat the procedure: since both eliminations of cluster-type vertices and of 1-dipoles reduce the number of vertices, the whole process ends in a finite number of steps, yielding a cluster-less cristallization of M^3.

The second part of the statement is a direct consequence of the second part of Proposition 2: if M^3 is handle-free and (Γ, γ) is any gem of M^3, both constructions of a rigid crystallization and of a cluster-less crystallization subsequently reduce the number of vertices, and hence the iteration of both processes necessarily yields a rigid cluster-less cristallization of M^3.

$\square$

Proposition 11 *Every closed connected 3-manifold M^3 admits a rigid cluster-less crystallization.*

Proof.

Let us assume $M^3 = J\#_m H$, $\#_m H$ being the connected sum of $m \geq 0$ copies of either the orientable or non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ and J being a closed handle-free 3-manifold. The statement may be easily proved by making use of the following fundamental facts:

- A rigid cluster-less crystallization of J may be obtained directly by means of Proposition 10, applied to any gem of J.

- There exists a rigid cluster-less crystallization (Ω, ω) (resp. $(\tilde{\Omega}, \tilde{\omega})$) of the orientable (resp. non-orientable) $\mathbb{S}^2$-bundle over $\mathbb{S}^1$: see Figure 6(a) (resp. Figure 6(b)).

- If (Γ_1, γ_1) (resp. (Γ_2, γ_2)) is a rigid cluster-less crystallization of M_1^3 (resp. M_2^3), then the "connected sum" $\Gamma_1\#\Gamma_2$ is a rigid cluster-less crystallization of $M_1^3\#M_2^3$. In fact, $\Gamma_1\#\Gamma_2$ is obviously a rigid crystallization (see [11], proof of Proposition 4), and the absence of cluster-type vertices may be proved easily by definition of graph connected sum.

$\square$

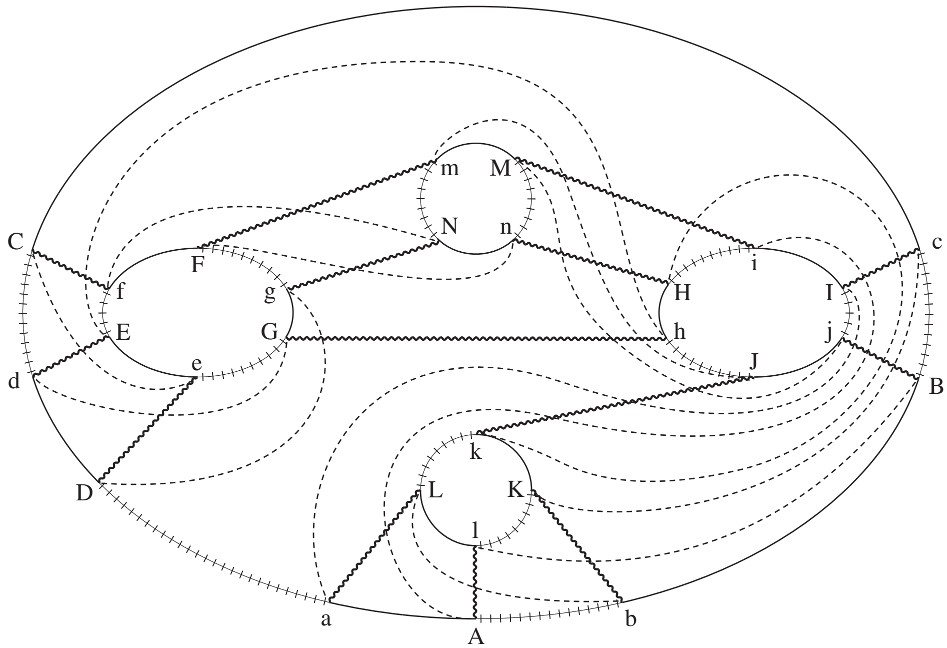

Figure 6(a)

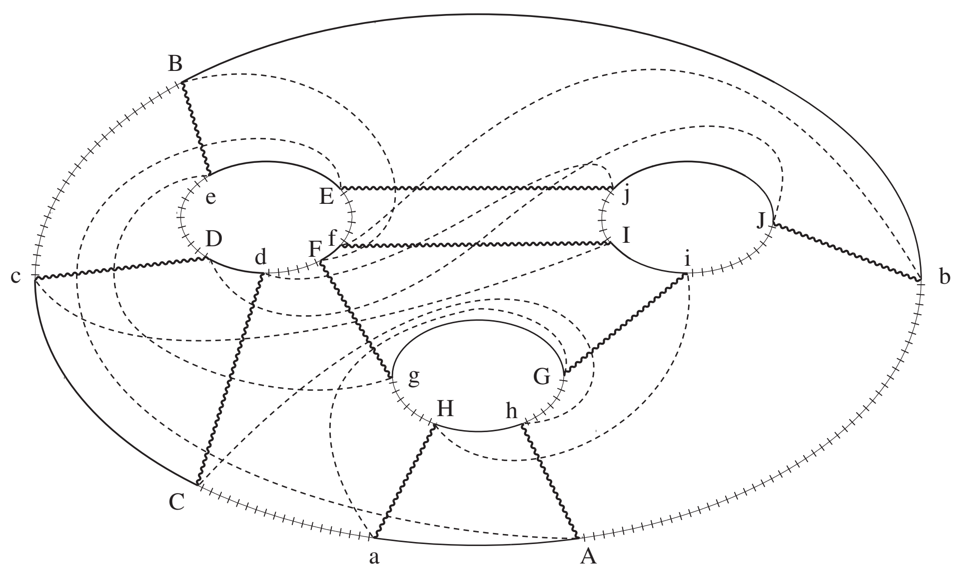

Figure 6(b)

Proposition 11 suggests naturally the construction of new catalogues $\mathcal{C}'^{(2p)}$ (resp. $\tilde{\mathcal{C}}'^{(2p)}$) containing rigid bipartite (resp. non bipartite) cluster-less crystallizations with $2p$ vertices: in fact, for increasing value of p, they yield an exhaustive representations of *all* orientable (resp. non-orientable) closed 3-manifolds.

The following table enables to compare the cardinality of catalogues $\mathcal{C}'^{(2p)}$ and $\mathcal{C}^{(2p)}$ (resp. $\tilde{\mathcal{C}}'^{(2p)}$ and $\tilde{\mathcal{C}}^{(2p)}$), for $1 \leq p \leq 15$. Notwithstanding the significative cut in

the number of involved crystallizations, Proposition 10 ensures that, if the attention is restricted to the handle-free case (as it obviously happens, in particular, when prime 3-manifolds are considered), no 3-manifold represented by $\mathbf{C}^{(2\bar{p})} = \bigcup_{1 \le p \le \bar{p}} \mathcal{C}^{(2p)}$ (resp. $\tilde{\mathbf{C}}^{(2\bar{p})} = \bigcup_{1 \le p \le \bar{p}} \tilde{\mathcal{C}}^{(2p)}$), for a fixed $\bar{p}$, is lost, when restricting to $\mathbf{C}'^{(2\bar{p})} = \bigcup_{1 \le p \le \bar{p}} \mathcal{C}'^{(2p)}$ (resp. $\tilde{\mathbf{C}}'^{(2\bar{p})} = \bigcup_{1 \le p \le \bar{p}} \tilde{\mathcal{C}}'^{(2p)}$).

2p	$\#\mathcal{C}^{(2p)}$	$\#\mathcal{C}'^{(2p)}$	$\#\tilde{\mathcal{C}}^{(2p)}$	$\#\tilde{\mathcal{C}}'^{(2p)}$
2	1	1	0	0
4	0	0	0	0
6	0	0	0	0
8	1	1	0	0
10	0	0	0	0
12	1	1	0	0
14	1	1	1	0
16	3	3	1	1
18	4	2	1	0
20	23	16	9	2
22	44	20	12	4
24	262	114	88	17
26	1252	382	480	99
28	7760	1981	2790	494
30	56912	10921	21804	2989

Table 3: rigid and rigid cluster-less crystallizations up to 30 vertices.

The algorithm described in section 3, applied to catalogue $\mathbf{C}'^{(30)} = \bigcup_{1 \le p \le 15} \mathcal{C}'^{(2p)}$, yields exactly 41 classes representing the prime orientable 3-manifolds described in Theorem I, 69 classes representing all prime 3-manifolds already contained in $\mathbf{C}^{(28)} = \bigcup_{1 \le p \le 14} \mathcal{C}^{(2p)}$ and 63 classes representing non-trivial connected sums.

The bijective correspondence between classes of crystallizations and manifolds holds in all cases except for connected sums, more precisely, for manifolds $L(2,1)\#L(2,1)\# L(2,1)$, $L(2,1)\#L(2,1)\#L(2,1)\#L(2,1)$ and $L(2,1)\#L(2,1)\#L(2,1)\#L(3,1)$. On the other hand they are easily recognized through the connected sum program.

Furthermore, manifolds $L(4,1)\#(S^1 \times S^2)$, $L(5,2)\#(S^1 \times S^2)$, $S^3/Q_8\#(S^1 \times S^2)$ and $L(2,1)\#L(3,1)\#(S^1 \times S^2))$, already appearing in catalogue $\mathbf{C}^{(30)}$, turn out to have no cluster-less crystallization up to 30 vertices.

However, Proposition 11 ensures that these manifolds will appear in successive catalogues $\mathcal{C}'^{(2p)}$, with $p > 15$.

Hence, the generation and analysis of catalogue $\mathbf{C}'^{(30)} = \bigcup_{1 \le p \le 15} \mathcal{C}'^{(2p)}$ yield an alternative (and more efficient [15]) procedure to prove the statement of Theorem I, together

[15] On the other hand, note that the procedure described in paragraph 3 turns out to be more useful to identify the manifolds represented by *all* gems up to 30 vertices: see paragraph 4.

with the existing results collected in Proposition 4 (see [25] and [15]).

We hope the catalogues $\mathcal{C}'^{(2p)}$ (resp. $\tilde{\mathcal{C}}'^{(2p)}$) can be useful to classify and recognize topologically closed orientable (resp. non-orientable) 3-manifolds admitting coloured triangulations with $2p$ tetrahedra, with $p > 15$.

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